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NON-LINEAR THEORY FOR THE DEFORMATION OF PRE-STRESSED CIRCULAR PLATES AND RINGS

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1. Introduction

It has been conjectured by Professor J.J. Stoker that solutions to non-linear elasticity problems are generally smoother in some sense than solutions to the corresponding linear problems. More specifically, if the solution to the linear problem has a singularity, the conjecture is that the solution to the non-linear problem will in many cases not have a singularity. Professor Stoker suggested that pre-stressed circular plates and rings would be examples of this phenomenon. It was this suggestion which led to the investigation presented here.

It is known that in the linear theory under the assumption of plane stress if a circular plate or ring is pre-stressed by slitting it and inserting a wedge or by deleting a wedge and welding the ends together, then the stress normal to a plane through the axis has a singularity at the center in the case of the plate, and in the case of the ring this stress on the inner curved lateral surface goes to infinity as the inner radius goes to zero. This is presented for the plate in [1], and it can be seen in the case of the ring from [2].

Since plane strain problems are more convenient in the non-linear theory than plane stress problems, we make here a comparison between the plane strain results of the two theories. In addition we derive the circular plate and ring into which the pre-stressed circular plate and ring respectively, deform when hydrostatic pressures are applied to their curved lateral surfaces. In the case of the ring different hydrostatic pressures are permitted on the two curved lateral surfaces. We also observe how the stress normal to a plane through the axis as computed from the non-linear theory reduces to that of the linear theory when the pre-stressing and applied hydrostatic pressures are small.

In addition to imposing the equilibrium equations and using the stress-strain laws to obtain boundary conditions, we require that the Jacobian of the deformation is positive, except at isolated points or curves, for both the linear and non-linear treatments. This is a natural condition in the non-linear theory, but it is usually omitted in the linear theory since in deriving the linear theory it is assumed that the displacements and their derivitives are small, making the condition superfluous. However, for the linear problems considered here, there are arbitrarily small values of certain strain and stress parameters for which the Jacobian is not positive for some values of other parameters. This can happen since the assumption of small derivatives of displacements is violated for some values of the parameters in the problem. Hence the positive Jacobian condition is not superfluous in the linear treatment of these problems.

For the pre-stressed circular plate it is shown that the stress normal to a plane through the axis is finite, although large, at the center when the non-linear theory is used. When the linear theory is used, there does not exist a pre-stressed plate of the type considered here obtained by deleting a wedge, but, when the plate is pre-stressed by inserting a wedge, the stress under consideration is infinite at the center of the plate. Thus the conjecture is confirmed in this instance.

For the type of pre-stressed circular ring considered here letting the inside radius go to zero after the pre-stressing is fixed is not permitted in the non-linear theory if the ring is pre-stressed by inserting a wedge, and it is not permitted in any case in the linear theory. It is the positive Jacobian condition which prohibits letting the inside radius go to zero. Hence no comparison of limits for the two theories is possible for any fixed pre-stressing. However, in the non-linear theory when the limit of the stress under consideration can be taken, it is finite.

It is shown that the discrepancy between the linear and non-linear theory results is due to the approximation $r^{\varepsilon} = 1 + \varepsilon \log r$. If $r_2 \geq r \geq r_1 > 0$ for some constants r_1, r_2 the approximation is arbitrarily good for $|\varepsilon|$ small enough. However, if ε is fixed, the approximation becomes arbitrarily bad as $r \longrightarrow 0$ no matter how small is the value of $|\varepsilon|$.

A special strain energy density is used in deriving the non-linear theory results. No approximations are made in the non-linear theory. The material described by this strain energy density function is homogeneous and isotropic, and the strain energy density agrees for small strains with that of the linear theory.

2. The Non-Linear Theory

The notation used here is that of Fritz John [3]. We simply list the desired results without proof. Consider a fixed rectangular Cartesian reference frame X. A particle at the point (x_1,x_2,x_3) in the undeformed body goes to a point $(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the deformed body. We think of $\overline{x}_i$ as a function of the variables x_j and let

$$(2.1) \qquad p_{ij} = \frac{\partial \overline{x}_i}{\partial x_j} .$$

Let W be the strain energy density function of the material. W is taken to be a function of the variables x_i and p_{ij}, and we let

$$(2.2) \qquad q_{ij} = \frac{\partial W}{\partial p_{ij}} .$$

Then from [3] the equilibrium equations are

$$(2.3) \qquad \frac{\partial q_{ij}}{\partial x_j} = 0$$

if the body forces are zero. The usual sumation convention is used.

Let $\vec{t} = (t_1, t_2, t_3)$ be the surface traction vector acting on the boundary of the body where t_i is given in terms of force per unit deformed area. Then from [3]

$$(2.4) \qquad t_i = q_{ij} n_j \frac{dS}{d\bar{S}}$$

where (n_1, n_2, n_3) is the unit outer normal vector to the undeformed boundary and dS and $d\bar{S}$ are the elements of area on the undeformed and deformed boundaries respectively.

Let p be the matrix (p_{ij}), and let c be the unique rotation matrix such that c^*p is symmetric and positive definite (* denotes the transpose). Such a rotation matrix c exists if $\det p > 0$ which we now assume. Then c is called the local rotation matrix since c rotates the directions of principal extension in the undeformed body into those in the deformed body (see discussion of c in [4]). Hence we list

(2.5) $c^*c = 1$, $\det c = 1$, c^*p is symmetric and positive definite.

It can be shown that $c^*p = \sqrt{p^*p}$, the symmetric positive definite square root matrix of p^*p. For the strain matrix η we use

$$(2.6) \qquad \eta = c^*p - 1 = \sqrt{p^*p} - 1 .$$

Then the eigenvalues of η are the stationary values of $d\bar{s}/ds - 1$ where ds and $d\bar{s}$ are arc length in the undeformed and deformed bodies respectively.

For the strain energy density function W we choose

$$(2.7) \qquad W = \frac{\lambda}{2}[\eta]^2 + \mu[\eta^2]$$

where λ and μ are the Lamé constants and the square bracket denotes the trace of the matrix. W agrees for small strains with the strain energy density of the linear theory for a homogeneous isotropic material.

From [4] we have

$$(2.8) \qquad q_{ij} = (\lambda[\eta] - 2\mu)c_{ij} + 2\mu p_{ij} .$$

3. Tensor Formulation

Since we will be working in curvilinear coordinates (cylindrical coordinates), we introduce tensor methods for convenience. Let $\theta_1, \theta_2, \theta_3$ be curvilinear coordinates such that $\det(\partial x_i/\partial\theta_j) > 0$. In the table below the left and right columns present the notation used for the X-components and θ-compoents of the same covarient tensor.

X-components	θ-components
δ_{ij}	g_{ij}
$\overline{x}_i$	u_i
p_{ij}	P_{ij}
c_{ij}	C_{ij}
η_{ij}	E_{ij}
q_{ij}	Q_{ij}
t_i	T_i
n_i	N_i

We define the quantities g^{ij} by $(g^{ij}) = (g_{ij})^{-1}$ and use the quantities g_{ij} and g^{ij} to lower and raise the indices of the θ-components of tensors in the usual way (i.e. $P^i{}_j = g^{ik}P_{kj} = g_{jk}P^{ik}$, etc.). Then the results of section 2 become the following:

(3.1) $$P_{ij} = u_i|_j$$

where $|_j$ denotes covarient differentiation with respect to θ_j using the quantities g_{ij} as the components of the metric tensor

(3.2) $$Q_{ij} = \frac{\partial W}{\partial P^{ij}}$$

(3.3) $$Q^{ij}|_j = 0$$

(3.4) $$T^i = Q^{ij} N_j \frac{dS}{d\bar{S}}$$

(3.5) $C_{ki} C^{kj} = \delta^j_i$, det $(C_{ij}) > 0$, $(C_{ki} P^k{}_j)$ is symmetric and positive definite

(3.6) $$E_{ij} = C_{ki} P^k{}_j - g_{ij}$$

(3.7) $$W = \frac{\lambda}{2} E^i_i E^j_j + \mu E^i_j E^j_i$$

(3.8) $$Q^{ij} = (\lambda E^k_k - 2\mu) C^{ij} + 2\mu P^{ij} .$$

4. The Pre-Stressed Ring under Hydrostatic Pressures (Non-Linear)

It will be convenient to use cylindrical coordinates r, θ, Z given by $x_1 = r \cos\theta$, $x_2 = r \sin\theta$, $x_3 = Z$. At first consider the part of a circular ring which in the undeformed state occupies the region $r_1 \leq r \leq r_2$, $\alpha_1 \leq \theta \leq \alpha_2$, $|Z| \leq h$. We ask if for each choice of the constants $\varepsilon > -1$, P_1 , P_2 , r_1 , r_2 , there is a function $f(r)$ independent of α_1 and α_2 such that the deformation

(4.1) $\bar{x}_1 = f(r) \cos(1+\varepsilon)\theta$, $\bar{x}_2 = f(r) \sin(1+\varepsilon)\theta$, $\bar{x}_3 = Z$

has positive Jacobian and satisfies the equilibrium equations (3.3) with hydrostatic pressures of $2\mu P_1$ and $2\mu P_2$ applied to the boundaries $r = r_1, r_2$ respectively. The units of $2\mu P_1$ and $2\mu P_2$ are force per unit deformed area. We will show that the answer to this question is yes, if P_1, P_2, ε, and r_1/r_2 are restricted suitably, and that the force distribution on the surfaces $\theta = \alpha_1, \alpha_2$ is normal to the deformed surfaces. Hence when $\varepsilon < 0$ we can obtain a stressed ring with hydrostatic pressures of $2\mu P_1$ and $2\mu P_2$ on the inner and outer curved lateral surfaces respectively by welding together two such pieces. Suitable pieces would be one for which $\alpha_1 = 0, \alpha_2 = 2\pi$ and one for which $\alpha_1 = 2\pi$, $\alpha_2 = 2\pi/1+\varepsilon$. Also when $\varepsilon > 0$ a stressed ring with hydrostatic pressures of $2\mu P_1$ and $2\mu P_2$ on the inner and outer curved lateral surfaces respectively can be obtained by welding together the ends of the piece for which $\alpha_1 = 0$, $\alpha_2 = 2\pi/1+\varepsilon$. If we take $P_1 = P_2 = 0$, we obtain what we are calling a pre-stressed ring. If we take $P_1 \neq 0$ or $P_2 \neq 0$, we obtain the ring into which the pre-stressed ring will deform under hydrostatic pressures $2\mu P_1$ and $2\mu P_2$.

Before proceding with the calculation, we outline the work. First the positive Jacobian condition is (4.1a). Substituting the deformation (4.1) into the three equilibrium equations of the non-linear theory, we obtain one ordinary second order differential equation for f . The general solution of this equation is linear in the constants of integration A and B (see (4.15)).

Using the stress-strain laws of the non-linear theory, the condition that hydrostatic pressures of $2\mu P_1$ and $2\mu P_2$ are applied on the curved lateral boundaries reduces to two linear equations, (4.21), in A and B. We want the determinant D of this system of equations to be non-zero in order that A and B are determined uniquely. Since $D > 0$ when $\varepsilon = P_1 = P_2 = 0$ and $r_2 > r_1$, we make the requirement $D > 0$ for all values of $\varepsilon, P_1, P_2, r_1$, and r_2 considered. This is (4.23).

Assuming $D > 0$, then (4.1a) implies (4.25) and (4.26) where $k = r_1/r_2$. Conversely, $D > 0$, (4.25), and (4.26) all together imply that the Jacobian is positive. These conditions are summarized in (4.27), and, when the parameters ε, P_1, P_2, and k satisfy (4.27), there is a unique f such that the deformation (4.1) satisfies all the conditions we have imposed.

We want to be able to build the pre-stressed ring before we apply the hydrostatic pressures to the curved lateral surfaces. That is, we want (4.27) to be satisfied when $P_1 = P_2 = 0$. This gives (4.28) as a new restriction on ε and k. (4.28) is always satisfied if $\varepsilon > 0$, i.e. if the ring is pre-stressed by deleting a wedge. But if $-1 < \varepsilon < 0$, (4.28) is not satisfied unless k is near enough to one. That is, the ring can not be pre-stressed by inserting a wedge unless the ring is thin enough.

Once ε and k are chosen so that (4.28) is satisfied, then (4.27) becomes restrictions on the hydrostatic pressures only. For the values of P_1 and P_2 satisfying (4.27) there is a unique f such that (4.1) satisfies the equilibrium equations, the boundary conditions, and the positive Jacobian condition.

Since (4.28) is always satisfied when $\varepsilon > 0$ and $0 < k < 1$, we can take limits as $r_1 \longrightarrow 0$ in the pre-stressed ring if the pre-stressing is done by deleting a wedge. Since (4.28) is not satisfied when $-1 < \varepsilon < 0$ unless k is near enough to one, we can not take limits as $r_1 \longrightarrow 0$ in the pre-stressed ring if the pre-stressing is done by inserting a wedge. We do not discuss the problem of taking limits as $r_1 \longrightarrow 0$ if the pre-stressed ring is first loaded further by taking P_1 or $P_2 \neq 0$. That problem could be solved by further study of (4.27).

The surface traction vector acting across a plane through the axis of the ring is derived and is observed to be normal to the deformed plane. Its magnitude is $2\mu T$ where T is given by (4.30). It is shown that $\lim_{r_1 \to 0} T(r_1) = -1$ when $\varepsilon > 0$, so that this force has a finite limit when we can let $r_1 \longrightarrow 0$.

We now present the details of the work. The Jacobian of (4.1) is $\frac{1+\varepsilon}{r} ff'$. Since we are restricting ε so that $\varepsilon > -1$, the positive Jacobian condition implies f and f' are not zero and have the same sign. Since f is the distance from the axis of the deformed ring to a particle in the ring, we have $f \geq 0$. Hence the positive Jacobian condition becomes

(4.1a) $\quad f > 0, \quad f' > 0 \quad \text{for} \quad r_1 \leq r \leq r_2$.

We make the identification $\theta_1 = r, \theta_2 = \theta, \theta_3 = Z$. Then

$$\left(\frac{\partial x_i}{\partial \theta_j}\right) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(\frac{\partial \theta_i}{\partial x_j}\right) = \left(\frac{\partial x_i}{\partial \theta_j}\right)^{-1}$$

(4.2)

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $(g_{ij}) = \left(\frac{\partial x_i}{\partial \theta_j}\right)^* \left(\frac{\partial x_i}{\partial \theta_j}\right)$ and $(g^{ij}) = (g_{ij})^{-1}$,

(4.3) $$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g = r^2.$$

The non-zero Christoffel symbols are

(4.4) $$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^1_{22} = -r .$$

Hence

$$(4.5)\quad (P^i{}_j) = (u^i|_j) = \begin{pmatrix} \frac{\partial u^1}{\partial r} & \frac{\partial u^1}{\partial \theta} - ru^2 & \frac{\partial u^1}{\partial Z} \\ \frac{\partial u^2}{\partial r} + \frac{1}{r}u^2 & \frac{\partial u^2}{\partial \theta} + \frac{1}{r}u^1 & \frac{\partial u^2}{\partial Z} \\ \frac{\partial u^3}{\partial r} & \frac{\partial u^3}{\partial \theta} & \frac{\partial u^3}{\partial Z} \end{pmatrix}$$

and

$$(4.6)\quad \begin{cases} A^{1j}|_j = \frac{\partial A^{1j}}{\partial \theta_j} + \frac{1}{r}A^{11} - rA^{22} \\ A^{2j}|_j = \frac{\partial A^{2j}}{\partial \theta_j} + \frac{1}{r}(A^{12} + 2A^{21}) \\ A^{3j}|_j = \frac{\partial A^{3j}}{\partial \theta_j} + \frac{1}{r}A^{31} \end{cases}$$

for any contravarient tensor A^{ij} .

From (4.1) and (4.2)

$$(4.7)\qquad u^1 = f\cos\varepsilon\theta,\quad u^2 = \frac{f}{r}\sin\varepsilon\theta,\quad u^3 = Z\ .$$

From (4.5) and (4.7)

$$
(4.8)\quad \begin{cases}
(P^i{}_j) = \begin{pmatrix} f' \cos \varepsilon\theta & -(1+\varepsilon)\, f \sin \varepsilon\theta & 0 \\ \frac{f'}{r} \sin \varepsilon\theta & (1+\varepsilon)\, \frac{f}{r} \cos \varepsilon\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\[2ex]
(P^{ij}) = (P^i{}_k g^{kj}) = \begin{pmatrix} f' \cos \varepsilon\theta & -(1+\varepsilon)\, \frac{f}{r^2} \sin \varepsilon\theta & 0 \\ \frac{f'}{r} \sin \varepsilon\theta & (1+\varepsilon)\, \frac{f}{r^3} \cos \varepsilon\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{cases}
$$

For a plane strain deformation the rotation matrix c has the form

$$
c = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

where ψ is the local rotation angle. Using this with (4.2) we obtain

$$
(4.9)\qquad (c_{ij}) = \begin{pmatrix} \cos \psi & -r \sin \psi & 0 \\ r \sin \psi & r^2 \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

From (4.9) and (4.8)

$$(4.10)\quad (C_{ki}P^k{}_j) = \begin{pmatrix} f' \cos(\psi-\varepsilon\theta) & (1+\varepsilon)f \sin(\psi-\varepsilon\theta) & 0 \\ -rf' \sin(\psi-\varepsilon\theta) & (1+\varepsilon)rf \cos(\psi-\varepsilon\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since (3.5) determines the quantities C_{ij} uniquely, we see from (4.9) and (4.10) that it suffices to choose $\psi = \varepsilon\theta$. Then $(C_{ki}P^k{}_j)$ is symmetric and positive definite by (4.1a). Hence

$$(4.11)\quad \begin{cases} (C_{ij}) = \begin{pmatrix} \cos\varepsilon\theta & -r\sin\varepsilon\theta & 0 \\ r\sin\varepsilon\theta & r^2\cos\varepsilon\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\[2ex] (C^{ij}) = (g^{ik}g^{j\ell}C_{k\ell}) = \begin{pmatrix} \cos\varepsilon\theta & -\frac{1}{r}\sin\varepsilon\theta & 0 \\ \frac{1}{r}\sin\varepsilon\theta & \frac{1}{r^2}\cos\varepsilon\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{cases}$$

From (4.10) and (3.6)

$$(4.12)\quad \begin{cases} (E_{ij}) = \begin{pmatrix} f'-1 & 0 & 0 \\ 0 & (1+\varepsilon)rf-r^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\[2ex] E^k_k = f' + (1+\varepsilon)\frac{f}{r} - 2 \end{cases}$$

From (3.3) and (3.8) the equilibrium equations are

$$(4.13) \quad (\lambda E^k_k - 2\mu)C^{ij}|_j + 2\mu P^{ij}|_j + \lambda \frac{\partial E^k_k}{\partial \theta_j} C^{ij} = 0 .$$

From (4.6),(4.8), and (4.11)

$$(4.14) \quad \begin{cases} P^{1j}|_j = [f'' + \frac{1}{r} f' - (\frac{1+\varepsilon}{r})^2 f] \cos \varepsilon\theta \\ P^{2j}|_j = \frac{1}{r} [f'' + \frac{1}{r} f' - (\frac{1+\varepsilon}{r})^2 f] \sin \varepsilon\theta \\ P^{3j}|_j = 0 \\ C^{1j}|_j = - \frac{\varepsilon}{r} \cos \varepsilon\theta \\ C^{2j}|_j = - \frac{\varepsilon}{r^2} \sin \varepsilon\theta \\ C^{3j}|_j = 0 \end{cases}$$

Substituting into (4.13) from (4.12) and (4.14), only one differential equation for f is obtained, namely

$$f'' + \frac{1}{r} f' - (\frac{1+\varepsilon}{r})^2 f = - \frac{\varepsilon}{(1-\sigma)r}$$

where σ is Poisson's ratio.

The general solution of this equation is

$$(4.15) \qquad f = \frac{1}{(2+\varepsilon)(1-\sigma)} \left(r + Ar^{1+\varepsilon} + \frac{B}{r^{1+\varepsilon}}\right)$$

where A and B are constants of integration.

From (4.15) and (4.12)

$$(4.16)\quad \begin{cases} f' = \dfrac{1}{(2+\varepsilon)(1-\sigma)}\left[1 + (1+\varepsilon)\left(Ar^{\varepsilon} - \dfrac{B}{r^{2+\varepsilon}}\right)\right] \\[2ex] E_k^k = \dfrac{2A(1+\varepsilon)}{(2+\varepsilon)(1-\sigma)}\, r^{\varepsilon} - \dfrac{1-2\sigma}{1-\sigma} \end{cases}$$

On the boundaries $r = r_1, r_2$ we have $d\overline{S} = (1+\varepsilon) f d\theta dZ$ and $dS = r d\theta dZ$. On $r = r_2$ we have $n_1 = \cos\theta, n_2 = \sin\theta$, $n_3 = 0$ so that $N_1 = 1, N_2 = N_3 = 0$. Similarly on $r = r_1$, $N_1 = -1, N_2 = N_3 = 0$. Therefore from (3.4)

$$(4.17)\quad \begin{cases} T^1 = -\dfrac{r}{(1+\varepsilon)f}\, Q^{11} \quad \text{for} \quad r = r_1 \\[2ex] T^1 = \dfrac{r}{(1+\varepsilon)f}\, Q^{11} \quad \text{for} \quad r = r_2 \ . \end{cases}$$

The unit outer normal vectors to the deformed boundaries $r = r_1, r_2$ are $\mp(\cos(1+\varepsilon)\theta, \sin(1+\varepsilon)\theta, 0)$ respectively. Hence the requirement that hydrostatic pressures of $2\mu P_1$ and $2\mu P_2$ act on $r = r_1, r_2$ respectively are
$(t_1, t_2, t_3) = 2\mu P_1(\cos(1+\varepsilon)\theta, \sin(1+\varepsilon)\theta, 0)$ for $r = r_1$ and
$(t_1, t_2, t_3) = -2\mu P_2(\cos(1+\varepsilon)\theta, \sin(1+\varepsilon)\theta, 0)$ for $r = r_2$.
From these conditions and (4.2)

$$(4.18)\quad \begin{cases} \left.\begin{aligned} T^1 &= 2\mu P_1 \cos \varepsilon\theta \\ T^2 &= \frac{2\mu P_1}{r} \sin \varepsilon\theta \\ T^3 &= 0 \end{aligned}\right\} & \text{for} \quad r = r_1 \\ \left.\begin{aligned} T^1 &= -2\mu P_2 \cos \varepsilon\theta \\ T^2 &= \frac{-2\mu P_2}{r} \sin \varepsilon\theta \\ T^3 &= 0 \end{aligned}\right\} & \text{for} \quad r = r_2 \end{cases}$$

Thus (4.17) and (4.18) together with (4.15) give

$$(4.19)\quad \left.\begin{cases} Q^{11} = -\dfrac{E(1+\varepsilon)P_\alpha}{(2+\varepsilon)(1-\sigma^2)}\left(1+Ar^{\varepsilon}+\dfrac{B}{r^{2+\varepsilon}}\right)\cos \varepsilon\theta \\ Q^{21} = -\dfrac{E(1+\varepsilon)P_\alpha}{(2+\varepsilon)(1-\sigma^2)}\left(1+Ar^{\varepsilon}+\dfrac{B}{r^{2+\varepsilon}}\right)\dfrac{1}{r}\sin \varepsilon\theta \\ Q^{31} = 0 \end{cases}\right\}\text{for } r=r_\alpha (\alpha=1,2)$$

where E is Young's modulus.

From (3.8),(4.8),(4.11), and (4.16)

$$(4.20)\quad \begin{cases} Q^{11} = \dfrac{E(1+\varepsilon)}{(2+\varepsilon)(1-\sigma^2)}\left[-1+\dfrac{A}{1-2\sigma}\,r^{\varepsilon}-\dfrac{B}{r^{2+\varepsilon}}\right]\cos\varepsilon\theta \\[2ex] Q^{21} = \dfrac{E(1+\varepsilon)}{(2+\varepsilon)(1-\sigma^2)}\left[-1+\dfrac{A}{1-2\sigma}\,r^{\varepsilon}-\dfrac{B}{r^{2+\varepsilon}}\right]\dfrac{1}{r}\sin\varepsilon\theta \\[2ex] Q^{31} = 0 \end{cases}$$

Hence (4.19) and (4.20) give

$$(4.21)\quad \left(P_\alpha + \frac{1}{1-2\sigma}\right)Ar_\alpha^{\varepsilon} - \frac{(1-P_\alpha)B}{r_\alpha^{2+\varepsilon}} = 1 - P_\alpha \quad \text{for} \quad \alpha = 1,2 \ .$$

The determinant D of coefficients of the system of equations (4.21) (considering A and B as unknowns) is

$$(4.22)\quad \begin{aligned} D = \left(\frac{1}{r_1 r_2}\right)^{2+\varepsilon} \Big[&\left(P_2 + \frac{1}{1-2\sigma}\right)(1-P_1)r_2^{2+2\varepsilon} \\ &- \left(P_1 + \frac{1}{1-2\sigma}\right)(1-P_2)r_1^{2+2\varepsilon}\Big] \ . \end{aligned}$$

In order that A and B are uniquely determined we require $D \neq 0$. When $\varepsilon = P_1 = P_2 = 0, D > 0$ since $r_2 > r_1$. In order that we can deform continuously from the unstressed ring to the stressed ring having A and B always uniquely determined, we must therefore require that $D > 0$ for all $P_1, P_2, \varepsilon > -1$, and $\dfrac{r_1}{r_2}$ considered. This restriction is

$$(4.23)\quad \left(P_2 + \frac{1}{1-2\sigma}\right)(1-P_1) > \left(P_1 + \frac{1}{1-2\sigma}\right)(1-P_2)\left(\frac{r_1}{r_2}\right)^{2+2\varepsilon} \ .$$

From (4.21)

$$(4.24)\quad \begin{cases} A = \frac{1}{D}(1-P_1)(1-P_2)\left(\frac{1}{r_1^{2+\varepsilon}} - \frac{1}{r_2^{2+\varepsilon}}\right) \\ \\ B = \frac{1}{D}\left[(P_1+\frac{1}{1-2\sigma})(1-P_2)r_1^{\varepsilon}-(P_2+\frac{1}{1-2\sigma})(1-P_1)r_2^{\varepsilon}\right] \end{cases} .$$

We have yet to impose the conditions (4.1a). From (4.15) and (4.24) $(2+\varepsilon)(1-\sigma)r_1 f(r_1) = r_1^2+Ar_1^{2+\varepsilon}+\frac{B}{r_1{}^{\varepsilon}}$ $= \frac{2}{D}\,\frac{1-\sigma}{1-2\sigma}(1-P_2)\left[1-\left(\frac{r_1}{r_2}\right)^{2+\varepsilon}\right]$, and $(2+\varepsilon)(1-\sigma)r_2 f(r_2)$ $= \frac{2}{D}\,\frac{1-\sigma}{1-2\sigma}(1-P_1)\left[\left(\frac{r_2}{r_1}\right)^{2+\varepsilon}-1\right]$. Since $\varepsilon > -1, 0 < \sigma < \frac{1}{2}$, $r_1 < r_2$, and $D > 0$, then $f(r) > 0$ for $r_1 \leq r \leq r_2$ implies the restrictions

$$(4.25)\qquad P_1 < 1\ ,\quad P_2 < 1\ .$$

Let $k = \frac{r_1}{r_2}$ so that $0 < k < 1$. From (4.16),(4.22) and (4.24) $(2+\varepsilon)(1-\sigma)r_1^2 f'(r_1) = r_1^2+(1+\varepsilon)(Ar_1^{2+\varepsilon}-\frac{B}{r_1^{\varepsilon}})$

$= \frac{1}{D}\Big\{(P_2+\frac{1}{1-2\sigma})(1-P_1)(2+\varepsilon)k^{-\varepsilon}-(P_1+\frac{1}{1-2\sigma})(1-P_2)(k^{2+\varepsilon}+1+\varepsilon)$ $+(1+\varepsilon)(1-P_1)(1-P_2)(1-k^{2+\varepsilon})\Big\}$. Hence $f'(r_1) > 0$ implies

$$(4.26)\quad \begin{aligned} &(P_2+\frac{1}{1-2\sigma})(1-P_1)(2+\varepsilon)k^{-\varepsilon}-(P_1+\frac{1}{1-2\sigma})(1-P_2)(k^{2+\varepsilon}+1+\varepsilon) \\ &\quad +(1+\varepsilon)(1-P_1)(1-P_2)(1-k^{2+\varepsilon}) > 0 \ . \end{aligned}$$

So far we have shown that (4.25) and (4.26) are necessary for (4.1a) when (4.23) is imposed. We now show that (4.23), (4.25), and (4.26) are sufficient. Inequality (4.26) gives $f'(r_1) > 0$ as already shown. But (4.23),(4.24), and (4.25) imply $A > 0$ so that $f'(r_1) > 0$ and (4.16) gives

$$B < \frac{r_1^{2+\varepsilon}}{1+\varepsilon} + Ar_1^{2+2\varepsilon} \leq \frac{r^{2+\varepsilon}}{1+\varepsilon} + Ar^{2+2\varepsilon}, 1+(1+\varepsilon)\left(Ar^{\varepsilon}-\frac{B}{r^{2+\varepsilon}}\right) > 0 \ ,$$

and $f'(r) > 0$ all for $r_1 \leq r \leq r_2$. We have already restricted P_2 so that $f(r_1) > 0$. This with $f'(r) > 0$ implies that $f(r) > 0$ for $r_1 < r < r_2$.

Using (4.25), we can rewrite (4.23),(4.25), and (4.26) as

$$(4.27)\left\{\begin{aligned} &P_1 < 1, \quad P_2 < 1 \\ &\frac{P_2+\frac{1}{1-2\sigma}}{1-P_2} > \frac{P_1+\frac{1}{1-2\sigma}}{1-P_1}\, k^{2+2\varepsilon} \\ &\frac{P_2+\frac{1}{1-2\sigma}}{1-P_2}\,(2+\varepsilon)k^{-\varepsilon} > \frac{P_1+\frac{1}{1-2\sigma}}{1-P_1}\,(k^{2+\varepsilon}+1+\varepsilon)-(1+\varepsilon)(1-k^{2+\varepsilon}). \end{aligned}\right.$$

In order that we can pre-stress the ring before applying hydrostatic pressures, we want (4.27) to be valid for $P_1 = P_2 = 0$. This gives the additional restriction

$$(4.28)\quad (2+\varepsilon)k^{-\varepsilon} > [1+(1+\varepsilon)(1-2\sigma)]k^{2+\varepsilon}+2\sigma(1+\varepsilon) \ .$$

To see if (4.28) is trivial, we let
$g(k) = [1+(1+\varepsilon)(1-2\sigma)]k^{2+\varepsilon}+2\sigma(1+\varepsilon)-(2+\varepsilon)k^{-\varepsilon}$ for $0 < k < 1$, $\varepsilon > -1$. Then (4.28) becomes $g(k) < 0$. First consider the case $\varepsilon > 0$. Then $g'(k) = (2+\varepsilon)\{[1+(1+\varepsilon)(1-2\sigma)]k^{1+\varepsilon} +\varepsilon k^{-1-\varepsilon}\} > 0$ so that $g(k)$ is a monotonically increasing function of k . Since also $g(1-) = 0$, we have $g(k) < 0$ for $0 < k < 1, \varepsilon > 0$, and the restriction is automatically satisfied.

Now consider the case $-1 < \varepsilon < 0$. Then
$g''(k) = (2+\varepsilon)(1+\varepsilon)\{[1+(1+\varepsilon)(1-2\sigma)k^{\varepsilon}-\varepsilon k^{-2-\varepsilon}\} > 0$. Hence $g'(k)$ is a monotonically increasing function of k for $0 < k < 1$. Since $g'(0+) = -\infty$ and $g'(1-) = 2(2+\varepsilon)(1+\varepsilon)(1-\sigma) > 0$, there is a unique $\bar{k}$ such that $0 < \bar{k} < 1$ and $g'(k) < 0$ for $0 < k < \bar{k}, g'(\bar{k}) > 0$ for $\bar{k} < k < 1$. Hence $g(k)$ has a minimum at $k = \bar{k}$ and is monotone in the intervals $0 < k < \bar{k}, \bar{k} < k < 1$. Since $g(1-) = 0$ and $g(0+) = 2\sigma(1+\varepsilon) > 0$, there is a unique k^* such that $0 < k^* < \bar{k}$ and $g(k) > 0$ for $0 < k < k^*, g(k) < 0$ for $k^* < k < 1$.

Hence (4.28) is trivial for $\varepsilon > 0$, but when $-1 < \varepsilon < 0$, (4.28) is not satisfied if k is too close to zero. In other words our criteria always permit pre-stressing the ring by deleting a portion and welding the ends together; but they do not permit pre-stressing by adding a piece unless the ring is thin enough. Thus if $\varepsilon < 0$, we can not let $r_1 \longrightarrow 0$ after fixing ε. After fixing ε and k so that (4.28) is satisfied, we may regard (4.27) as restrictions on P_1 and P_2 alone.

Since $(P + \frac{1}{1-2\sigma})/(1-P)$ is a monotonically increasing function of P for $P < 1$, if $\overline{P}_1$ and $\overline{P}_2$ are any values of P_1 and P_2 satisfying (4.27), then $\overline{P}_1$ and $\hat{P}_2$ also satisfy (4.27) for all $\hat{P}_2$ such that $\overline{P}_2 \leq \hat{P}_2 < 1$. Also as $P_1 \longrightarrow 1-$, we must have $P_2 \longrightarrow 1-$ in order that P_1 and P_2 satisfy (4.27). Hence in the P_1P_2- plane, (4.27) is satisfied in a region of the type shaded in Figure 1.

This completes the proof that the transformation given by (4.1),(4.15), and (4.24) satisfies the equilibrium equations, boundary conditions on $r = r_1, r_2$, the positive Jacobian condition, and that it is possible to deform continuously from the case $P_1 = P_2 = 0$ to the case $P_1 \neq 0$ or $P_2 \neq 0$ holding ε fixed provided that $P_1, P_2, \varepsilon > -1$, and k satisfy (4.27) and (4.28).

Before proceding it is interesting to note what would be obtained if we had not required that it be possible to go from the unstressed ring to the stressed ring with f determined uniquely at each step. When $D = 0$, equations (4.21) do not have a solution unless $P_1 = P_2 = 1$ and in that case there is a one parameter continuum of solutions. When $D < 0$, A and B are given by (4.24), but now the conditions $f(r_1) > 0$, $f(r_2) > 0$ imply $P_1 > 1, P_2 > 1$. Hence in the P_1P_2- plane, the regions corresponding to solutions with $D \geq 0$ and $D \leq 0$ meet only at the point $P_1 = P_2 = 1$. Hence it is not possible to go from solutions with $P_1 < 1, P_2 < 1$ to other solutions varying P_1 and P_2 continuously unless we go through the

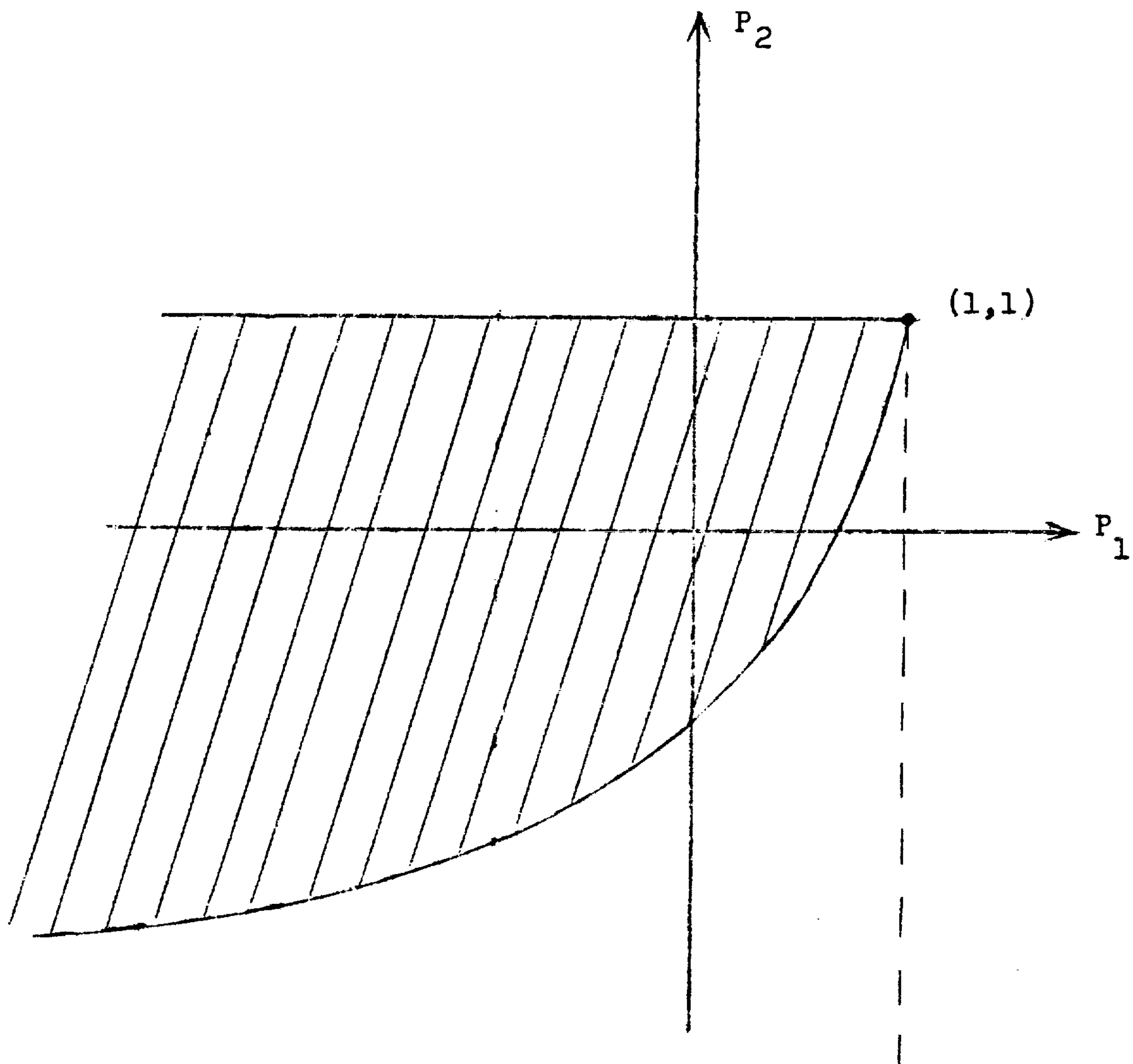

Figure 1

point $P_1 = P_2 = 1$ where the solution is not uniquely determined. The case $P_1 = P_2 = 1$ does not correspond to a buckled solution in the usual sense of the term since a continuum of solutions is possible when $P_1 = P_2 = 1$.

Proceding with the solutions we are admitting, we next show that the surface traction vector on $\theta = \alpha_1, \alpha_2$ is normal to the deformed surfaces. On $\theta = \alpha_2$ we have $dS = drdZ, d\overline{S} = f'drdZ$, and $\vec{n} = (-\sin\theta, \cos\theta, 0)$. Hence $N_1 = N_3 = 0, N_2 = r$, and $T^i = Q^{ij}N_j \frac{1}{f'} = \frac{r}{f'} Q^{i2}$ from (3.4). Using (3.8),(4.8),(4.11),(4.15), and (4.16), the preceding becomes

$$(4.29) \qquad \begin{cases} T^1 = -2\mu T \sin \varepsilon\theta \\ T^2 = 2\mu T \dfrac{\cos \varepsilon\theta}{r} \\ T^3 = 0 \end{cases}$$

where

$$(4.30) \qquad T(r) = \frac{\dfrac{A}{1-2\sigma} r^{\varepsilon} + \dfrac{B}{r^{2+\varepsilon}} - \dfrac{1}{1+\varepsilon}}{Ar^{\varepsilon} - \dfrac{B}{r^{2+\varepsilon}} + \dfrac{1}{1+\varepsilon}} .$$

Using (4.29) and (4.2), the surface traction vector $\vec{t}$ is $\vec{t} = 2\mu T(-\sin(1+\varepsilon)\theta, \cos(1+\varepsilon)\theta, 0) = 2\mu T\,\vec{n}$ where $\vec{n}$ is the unit outer normal vector to the deformed surface. Hence the surface traction vector is normal to the deformed boundary and $2\mu T$ is the normal component of stress on the deformed boundary $\theta = \alpha_2$.

A similar argument is valid for the boundary $\theta = \alpha_1$. We have yet to show that $\lim_{r_1 \to 0} T(r_1)$ is finite when the ring is pre-stressed by deleting a wedge (i.e. when $\varepsilon > 0$, $P_1 = P_2 = 0$). Actually we will show that $\lim_{r_1 \to 0} T(r_1)$ is finite whenever we can let $r_1 \longrightarrow 0$ while holding ε, P_1, and P_2 fixed. We have already shown that we can let $r_1 \longrightarrow 0$ while holding ε fixed positive and $P_1 = P_2 = 0$. Determining which other values of ε, P_1, and P_2 can be used in this limiting operation would require further study of the inequalities (4.27).

For convenience let

$$(4.31) \qquad C_\alpha = \frac{P_\alpha + \frac{1}{1-2\sigma}}{1-P_\alpha} \, , \; \alpha = 1,2 \quad .$$

From (4.22) and (4.24)

$$(4.32) \qquad \begin{cases} D = (C_2 - C_1 k^{2+2\varepsilon})(1-P_1)(1-P_2) \dfrac{r_2^{\varepsilon}}{r_1^{2+\varepsilon}} \\[2ex] A = \dfrac{1-k^{2+\varepsilon}}{C_2 - C_1 k^{2+2\varepsilon}} \dfrac{1}{r_2^{\varepsilon}} \\[2ex] B = \dfrac{C_1 k^{\varepsilon} - C_2}{C_2 - C_1 k^{2+2\varepsilon}} \; r_1^{2+\varepsilon} \quad . \end{cases}$$

Substituting into (4.30) from (4.32)

$$(4.33) \qquad T(r_1) = \frac{\frac{k^{\varepsilon}}{1-2\sigma}(1-k^{2+\varepsilon})+C_1k^{\varepsilon}-C_2-\frac{1}{1+\varepsilon}(C_2-C_1k^{2+2\varepsilon})}{k^{\varepsilon}(1-k^{2+\varepsilon})-C_1k^{\varepsilon}+C_2+\frac{1}{1+\varepsilon}(C_2-C_1k^{2+2\varepsilon})} \quad .$$

Hence

$$(4.34) \qquad \lim_{r_1 \to 0} T(r_1) = \begin{cases} \dfrac{1+C_1(1-2\sigma)}{(1-2\sigma)(1-C_1)} & \text{if } -1 < \varepsilon < 0 \\ -1 \quad \text{if } \varepsilon > 0 & . \end{cases}$$

Since the stress is $2\mu T$, we see that these limits are very large.

5. The Pre-Stressed Circular Plate under Hydrostatic Pressure (Non-Linear Theory).

Consider the part of a circular plate which in the undeformed state occupies the region $0 \leq r \leq R, \alpha_1 \leq \theta \leq \alpha_2, |Z| \leq h$. We ask if for each choice of the constants $\varepsilon > -1, P, R$ there is a function $f(r)$ independent of α_1 and α_2 such that the deformation (4.1) has positive Jacobian except at isolated points or curves and satisfies the equilibrium equations with a hydrostatic pressure of $2\mu P$ applied to the boundary $r = R$. Again the answer is yes under suitable restrictions, and the force vector on the surfaces $\theta = \alpha_1, \alpha_2$ is normal to the deformed surfaces so that stressed circular plates can be formed just as for the circular ring case.

The differential equation for f is the same as in the circular ring case so that f is given by (4.15). However, since in this case we want f to be finite for $r = 0$, we must take $B = 0$. Then (4.16) also remains valid for $B = 0$.

The consideration of boundary conditions is similar to that of the ring case and the equation for A is

$$(5.1) \qquad \left(P + \frac{1}{1-2\sigma}\right)AR^{\varepsilon} = 1-P \quad .$$

This has a solution only if $P \neq \frac{-1}{1-2\sigma}$. The requirement that we can go continuously from the case $\varepsilon \neq 0, P = 0$ to the case $\varepsilon \neq 0, P \neq 0$ by varying P continuously while holding ε fixed is

$$(5.2) \qquad P > \frac{-1}{1-2\sigma} \; .$$

Then

$$(5.3) \quad \begin{cases} A = \dfrac{1-P}{P+\dfrac{1}{1-2\sigma}} R^{-\varepsilon} \\[2ex] f(r) = \dfrac{r}{(2+\varepsilon)(1-\sigma)} \left[1 + \dfrac{1-P}{P+\dfrac{1}{1-2\sigma}} \left(\dfrac{r}{R}\right)^{\varepsilon} \right] \\[4ex] f'(r) = \dfrac{1+\varepsilon}{(2+\varepsilon)(1-\sigma)} \left[\dfrac{1}{1+\varepsilon} + \dfrac{1-P}{P+\dfrac{1}{1-2\sigma}} \left(\dfrac{r}{R}\right)^{\varepsilon} \right] \end{cases} \quad .$$

The positive Jacobian requirement is $f(r) > 0$ and $f'(r) > 0$ for $0 < r \leq R$ which gives

$$(5.4)\qquad \begin{cases} P < 1 & \text{for} \quad \varepsilon < 0 \\[2ex] \dfrac{1}{1+\varepsilon} + \dfrac{1-P}{P+\dfrac{1}{1-2\sigma}} > 0 & \text{for} \quad \varepsilon > 0 \; . \end{cases}$$

Hence it is always possible to go continuously from the pre-stressed plate to the pre-stressed plate under pressure $2\mu P$ by continuously changing P. That is for the circular plate we can always build the pre-stressed plate first and then load it by applying pressure.

Defining $T(r)$ as before we have $T(r) = \dfrac{\dfrac{A}{1-2\sigma} r^{\varepsilon} - \dfrac{1}{1+\varepsilon}}{Ar^{\varepsilon} + \dfrac{1}{1+\varepsilon}}$ by letting $B=0$ in (4.30). Hence

$$(5.5)\qquad \lim_{r \to 0} T(r) = \begin{cases} -1 & \text{if} \quad \varepsilon > 0 \\[2ex] \dfrac{1}{1-2\sigma} & \text{if} \quad -1 < \varepsilon < 0 \; . \end{cases}$$

In any case $\lim_{r \to 0} T(r)$ exists and is finite.

6. The Pre-Stressed Ring under Hydrostatic Pressures (Linear Theory).

Let $f = r + F$ and assume F and ε are so small their products can be neglected. Then (4.1) becomes

(6.1) $\overline{x}_1 = (r+F)\cos\theta - \varepsilon r\theta\sin\theta, \overline{x}_2 = (r+F)\sin\theta + \varepsilon r\theta\cos\theta, \overline{x}_3 = Z$.

We will show that F can be chosen so that (6.1) is an exact solution to the ring problem using the linear theory provided only that $|\varepsilon|, |P_1|$, and $|P_2|$ are small enough. The rectangular components of displacements are

(6.2) $\overline{x}_1 - x_1 = F\cos\theta - \varepsilon r\theta\sin\theta, \overline{x}_2 - x_2 = F\sin\theta + \varepsilon r\theta\cos\theta, \overline{x}_3 - x_3 = 0$.

Letting the quantities v^i be the cylindrical components of the contravarient displacement tensor, we have from (4.2) and (6.2)

(6.3) $$v^1 = F, v^2 = \varepsilon\theta, v^3 = 0 \quad .$$

Hence (4.5) and (4.3) imply

(6.4) $$(v^i|_j) = \begin{pmatrix} F' & -\varepsilon r\theta & 0 \\ \frac{\varepsilon\theta}{r} & \varepsilon + \frac{1}{r}F & 0 \\ 0 & 0 & 0 \end{pmatrix}, (v^i|^j) = \begin{pmatrix} F' & -\frac{\varepsilon\theta}{r} & 0 \\ \frac{\varepsilon\theta}{r} & \frac{\varepsilon}{r^2} + \frac{1}{r^3}F & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Let E^{ij} be the cylindrical components of the contra-varient linear strain tensor so that $E^{ij} = \frac{1}{2}(v^i|^j + v^j|^i)$. Then

$$(6.5) \qquad (E^{ij}) = \begin{pmatrix} F' & 0 & 0 \\ 0 & \frac{\varepsilon}{r^2} + \frac{1}{r^3} F & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The cylindrical components of the contravarient stress tensor are $\tau^{ij} = \lambda E^k_k g^{ij} + 2\mu E^{ij}$ so that the equilibrium equations $\tau^{ij}|_j = 0$ become

$$(6.6) \qquad \lambda \frac{\partial E^k_k}{\partial \theta_j} g^{ij} + 2\mu E^{ij}|_j = 0 .$$

But from (6.5) and (4.6)

$$(6.7) \qquad \begin{cases} E^k_k = F' + \frac{1}{r} F + \varepsilon \\ E^{1j}|_j = F'' + \frac{1}{r} F' - \frac{1}{r^2} F - \frac{\varepsilon}{r} \\ E^{2j}|_j = E^{3j}|_j = 0 . \end{cases}$$

Hence the only non-trivial equilibrium equation is $F'' + \frac{1}{r} F' - \frac{1}{r^2} F = \frac{1-2\sigma}{1-\sigma} \frac{\varepsilon}{r}$, and the general solution is

$$(6.8) \qquad F = \frac{1-2\sigma}{2(1-\sigma)} \varepsilon r \log r + Ar + \frac{B}{r} .$$

From (6.8)

$$(6.9)\quad \begin{cases} F' = \dfrac{1-2\sigma}{2(1-\sigma)}\,\varepsilon(\log r + 1) + A - \dfrac{B}{r^2} \\[2ex] E^k_k = \dfrac{1-2\sigma}{2(1-\sigma)}\,\varepsilon(2\log r + 1) + 2A + \varepsilon \\[2ex] \dfrac{1}{2\mu}\,\tau^{11} = \dfrac{\varepsilon}{2(1-\sigma)}\left(\log r + \dfrac{1-\sigma}{1-2\sigma}\right) + \dfrac{A}{1-2\sigma} - \dfrac{B}{r^2} \\[2ex] \tau^{21} = \tau^{31} = 0 \quad . \end{cases}$$

The boundary conditions are $\tau^{ij}N_j = -2\mu P_2 N^i$ for $r = r_2$ and $\tau^{ij}N_j = -2\mu P_1 N^i$ for $r = r_1$. Since $N_1 = 1, N_2 = N_3 = 0$ for $r = r_2$ and $N_1 = -1, N_2 = N_3 = 0$ for $r = r_1$, these become $\tau^{11} = -2\mu P_\alpha, \tau^{21} = \tau^{31} = 0$ for $r = r_\alpha (\alpha = 1,2)$. Hence

$$(6.10)\quad \frac{A}{1-2\sigma} - \frac{B}{r_\alpha^2} = \frac{-\varepsilon}{2(1-\sigma)}\left(\log r_\alpha + \frac{1-\sigma}{1-2\sigma}\right) - P_\alpha, \alpha = 1,2 \quad .$$

The unique solution to (6.10) is

$$(6.11)\quad \begin{cases} A = -\dfrac{\varepsilon}{2} + \dfrac{1-2\sigma}{2(1-\sigma)}\,\varepsilon\,\dfrac{k^2\log r_1 - \log r_2}{1-k^2} + \dfrac{(1-2\sigma)(k^2P_1 - P_2)}{1-k^2} \\[2ex] B = \dfrac{r_1^2}{1-k^2}\left[\dfrac{\varepsilon}{2(1-\sigma)}\log k + P_1 - P_2\right] . \end{cases}$$

The positive Jacobian condition is $(1+F')(1+\frac{F}{r}+\varepsilon)+\varepsilon^2\theta^2>0$. For this to be true for $\theta = 0$ we must have $(1+F')(1+\frac{F}{r}+\varepsilon) > 0$. This means that $1+F'$ and $1+\frac{F}{r}+\varepsilon$ are not zero and have the same sign. When r_1 and r_2 are fixed and $|\varepsilon|, |P_1|$, and $|P_2|$ are small enough, both quantities are positive. Since we want to admit all small enough $|\varepsilon|, |P_1|$, and $|P_2|$, we must require

$$(6.12)\qquad \begin{cases} 1 + \dfrac{F}{r} + \varepsilon > 0 \\ \\ 1 + F' > 0 \end{cases}$$

for all $\varepsilon, P_1, P_2, r_1,$ and r_2 admitted.

From (6.8), (6.9), and (6.11) we obtain

$$\frac{F(r_1)}{r_1} = -\frac{\varepsilon}{2} + \varepsilon\,\frac{\log k}{1-k^2} + \frac{(1-2\sigma)(k^2P_1-P_2)+P_1-P_2}{1-k^2}\,,$$

$$F'(r_1) = -\frac{\sigma}{1-\sigma}\,\varepsilon\left(\frac{1}{2} + \frac{\log k}{1-k^2}\right) + \frac{(1-2\sigma)(k^2P_1-P_2)-P_1+P_2}{1-k^2}\,.$$

Hence $\lim\limits_{r_1\to 0} \dfrac{F(r_1)}{r_1} = -\infty$ when $\varepsilon > 0$ and $\lim\limits_{r_1\to 0} F'(r_1) = -\infty$ when $\varepsilon < 0$. Hence (6.12) does not permit taking limits as $r_1 \longrightarrow 0$ when $\varepsilon, P_1, P_2,$ and r_2 are fixed, and we can not make comparisons with such limits taken in the non-linear theory.

On the surface $\theta = \alpha_2, N_1 = N_3 = 0$ and $N_2 = r$. Hence $T^i = \tau^{ij}N_j = r\tau^{i2}, T^1 = T^3 = 0, T^2 = r\tau^{22}$, and the surface traction vector is normal to the undeformed surface. Defining T by $T^i = 2\mu TN^i$, we have $T = \frac{r}{2\mu}T^2 = \frac{r^2}{2\mu}\tau^{22}$ and

$$(6.13)\qquad \begin{aligned} T(r) = {} & \frac{\varepsilon}{2(1-\sigma)}\left\{1+\log r + \frac{r^2(k^2\log r_1-\log r_2)+r_1^2\log k}{r^2(1-k^2)}\right\} \\ & + \frac{r^2(k^2P_1-P_2)+r_1^2(P_1-P_2)}{r^2(1-k^2)}\,. \end{aligned}$$

To conclude we observe that if $f-r, f'-1$, and $T(r)$, as computed from the non-linear theory in section 4, are expanded in power series in ε, P_1, and P_2, the terms up to first order give exactly the values of F, F', and $T(r)$ computed here with the linear theory. The important differences between the results of the two theories is due to the approximation $r^{\varepsilon} = 1 + \varepsilon \log r$. This approximation is arbitrarily good for $|\varepsilon|$ small enough provided that $0 < r_1 \leq r \leq r_2$ where r_1, r_2 are fixed; however, for ε fixed the approximation becomes arbitrarily bad as $r \longrightarrow 0$. Hence we do not expect agreement between the linear and non-linear theories when r_1 is small enough if r_2, ε, P_1, and P_2 are fixed.

7. The Pre-Stressed Circular Plate under Hydrostatic Pressure (Linear Theory).

Again we consider the deformation (6.1). The differential equation for F is the same as before so that (6.8) is the general solution. Requiring F to be finite for $r = 0$ gives $B = 0$. The boundary condition is (6.10) with $B = 0$ and r_α and P_α replaced by R and P. Hence

$$(7.1) \quad \begin{cases} A = -\dfrac{\varepsilon}{2} - \dfrac{1-2\sigma}{2(1-\sigma)}\, \varepsilon \log R - P(1-2\sigma) \\[2ex] F = r\left[-\dfrac{\varepsilon}{2} + \dfrac{1-2\sigma}{2(1-\sigma)}\, \varepsilon \log \dfrac{r}{R} - P(1-2\sigma)\right] \\[2ex] F' = -\dfrac{\sigma}{2(1-\sigma)}\, \varepsilon + \dfrac{1-2\sigma}{2(1-\sigma)}\, \varepsilon \log \dfrac{r}{R} - P(1-2\sigma) \end{cases}$$

Conditions (6.12) apply for $0 < r \leq R$. From (7.1)

$$1 + \frac{F}{r} + \varepsilon = 1 + \frac{\varepsilon}{2} + \frac{1-2\sigma}{2(1-\sigma)}\,\varepsilon \log\frac{r}{R} - P(1-2\sigma)\ ,$$

$$1 + F' = 1 - \frac{\sigma}{2(1-\sigma)}\,\varepsilon + \frac{1-2\sigma}{2(1-\sigma)}\,\varepsilon \log\frac{r}{R} - P(1-2\sigma)\ .$$

Since these are monotone functions of r, (6.12) is equivalent to requiring these expressions to be non-negative for $r = 0^-$ and positive for $r = R$. Hence

$$(7.2)\qquad \begin{cases} \varepsilon < 0 \\ \varepsilon < \dfrac{2(1-\sigma)}{\sigma}\,[1 - P(1-2\sigma)] \\ \varepsilon > 2[P(1-2\sigma)-1] \end{cases}$$

Thus the pre-stressing can not be done by deleting a wedge ($\varepsilon > 0$). Observe that whenever $|P|$ is small enough there exists ε values satisfying (7.2) .

For $T = \frac{r^2}{2\mu}\,\tau^{22} = \frac{r^2}{2\mu}\,[\lambda E^k_k g^{22} + 2\mu E^{22}]$ we have

$$(7.3)\qquad T = -P + \frac{\varepsilon}{2(1-\sigma)}\left(1 + \log\frac{r}{R}\right)\ .$$

Thus T has a singularity at the origin although it does not in the non-linear theory.

Again the results of the non-linear theory agree with the results of the linear theory to first order terms in ε and P, and the differences in the results of the two theories is due to the approximation $r^{\varepsilon} = 1 + \varepsilon \log r$.

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